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Statistical Entropy of a Schwarzschild Black Hole

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A model describing the internal microstates of particles is used to calculate the statistical entropy of a Schwarzschild black hole. The state of the system is described by a nonextensive entropy function which is superadditive and so fails to be concave. A strict maximum of the entropy does not exist; nonetheless, the entropy increases on merging two such systems.

KEY WORDS: Black-hole thermodynamics; concavity; nonextensivity; superadditivity.

1. INTRODUCTION

In a recent article,[1] nonextensive thermodynamic systems are considered and their relevance to black hole thermodynamics discussed. A function $f(\mathbf{X})$ is extensive if

$$f(a\mathbf{X}) = af(\mathbf{X}) \qquad (\mathbf{H}) \tag{1.1}$$

for any $a > 0$. For a gas, the variable $\mathbf{X} \equiv (U, V)$ with U the energy and V the volume, while for a Kerr–Newman black hole $\mathbf{X} \equiv (M, J, Q)$ with M the mass, J the angular momentum, and Q the electric charge. Black hole entropy is not extensive but strictly superadditive,[1] viz. the strict inequality in

$$S(\mathbf{X}_A + \mathbf{X}_B) \geqslant S(\mathbf{X}_A) + S(\mathbf{X}_B) \qquad (\mathbf{S}) \tag{1.2}$$

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is satisfied. In addition to criteria (1.1) and (1.2), the logical development[1] requires the notion of concavity

$$f(\lambda \mathbf{X}_A + (1-\lambda)\,\mathbf{X}_B) \geqslant \lambda f(\mathbf{X}_A) + (1-\lambda)\, f(\mathbf{X}_B) \qquad (\mathbf{C}) \tag{1.3}$$

for a constant λ, $0 \leqslant \lambda \leqslant 1$. The principles (1.1) to (1.3) are related by[1]

$$\mathbf{C} + \mathbf{S} \rightarrow \mathbf{H} \tag{1.4}$$

$$\mathbf{S} + \mathbf{H} \rightarrow \mathbf{C} \tag{1.5}$$

and

$$\mathbf{C} + \mathbf{H} \rightarrow \mathbf{S} \tag{1.6}$$

Consequently, if **S** is required[1] on physical grounds, then only two cases are possible (**S**, **H**, **C**) and (**S**, $\bar{\mathbf{H}}$, $\bar{\mathbf{C}}$). (The symbol $\bar{\mathbf{H}}$ denotes not **H**, or **H** is false, and so forth.) A black hole is not an ordinary thermodynamic system, i.e., of the type (**S**, **H**, **C**), in fact, it is of the type (**S**, $\bar{\mathbf{H}}$, $\bar{\mathbf{C}}$).

In what follows, we consider a collective-mode description of the interior of a Schwarzschild black hole, viz. $Q = J = 0$, and calculate the statistical entropy by counting the number of internal configurations of the black hole. Interestingly enough, our model is indeed of the type (**S**, $\bar{\mathbf{H}}$, $\bar{\mathbf{C}}$). Therefore, the entropy increases when two black holes coalesce but with no entropy maximization at the extremum. Our results are analogous to those for the generalized ideal gas of Ref. 1 with $h < g + 1$.

2. BLACK HOLE MODEL

Consider an ideal gas of distinguishable quasiparticles whose number is not conserved. The ith type of quasiparticles have mass m_i, momenta $\mathbf{p}$, and energies

$$E_i(\mathbf{p}) = (c^2 p^2 + m_i^2 c^4)^{1/2} \geqslant m_0 c^2 \tag{2.1}$$

where m_0 is the lowest mass of the system. The grand-canonical partition function with chemical potential $\mu_i = 0$ for all i is given by[2]

$$Z = \prod_i \left\{ 1 - \frac{V}{h^3} \int \exp[-E_i(\mathbf{p})/kT]\, d\mathbf{p} \right\}^{-1} \tag{2.2}$$

It should be remarked that the number of quasiparticles of each type is not conserved[2] owing to the creation and annihilation processes taking place and so $\mu_i = 0$ for all i. Note also that the quasiparticles are not ordinary particles, i.e., neither fermions nor bosons. In fact, the distinguishable quasiparticles possess the statistics which gives rise to the Gibbs paradox[2] in statistical mechanics.

The internal energy U is

$$U = -\left[\frac{\partial}{\partial(1/kT)} \ln Z\right]_V = \sum_i \frac{B_i(T)}{1 - C_i(T)} \tag{2.3}$$

where

$$B_i(T) \equiv \frac{V}{h^3} \int e^{-E_i(\mathbf{p})/kT} E_i(\mathbf{p})\, d\mathbf{p} = 3kTC_i(T) + \frac{4\pi}{c^3 h^3} (m_i c^2)^3 (kT)\, VK_1(m_i c^2/kT) \tag{2.4}$$

and the partition function for the ith type of quasiparticle is

$$C_i(T) \equiv \frac{V}{h^3} \int e^{-E_i(\mathbf{p})/kT}\, d\mathbf{p} = \frac{4\pi}{c^3 h^3} (m_i c^2)^2 (kT)\, VK_2(m_i c^2/kT) \leqslant 1 \tag{2.5}$$

with $K_\mu(x)$ the modified Bessel function of the second kind with subscript 2. The inequality in (2.5) is a consequence of the positiveness of the grand partition function (2.2). The entropy is given by

$$S = k\left[\frac{\partial}{\partial T}(T \ln Z)\right]_V = -k \sum_i \ln[1 - C_i(T)] + U/T \tag{2.6}$$

If the lowest mass $m_0 = 0$, then the inequality in (2.5) implies that for fixed V, $T \leqslant T_{\max}$ with[2]

$$VT_{\max}^3 = \pi^2(\hbar c/k)^3 \tag{2.7}$$

since $(d/dx)[x^2 K_2(x)] = -x^2 K_1(x) < 0$ for $0 \leqslant x < \infty$ and $z^\nu K_\nu(z) \to 2^{\nu-1}\Gamma(\nu)$ as $|z| \to 0$ for $\operatorname{Re} \nu > 0$. Note that the mean number of quasiparticles is[3]

$$\bar{n}_i(\mathbf{p}) = \frac{e^{-E_i(\mathbf{p})/kT}}{1 - C_i(T)} \tag{2.8}$$

Therefore, the single-particle distribution is of the Maxwell–Boltzmann type and so a maximum temperature for fixed V, viz., $C_i(T) \leqslant C_0(T) = (8\pi/c^3 h^3)(kT)^3\, V \leqslant 1$ with the aid of (2.5), implies $\bar{n}_i(\mathbf{p}) \geqslant 0$ for all i.

The pressure is given by

$$P = kT\left(\frac{\partial}{\partial V} \ln Z\right)_T = \frac{kT}{V} \sum_i \frac{C_i(T)}{1 - C_i(T)} \tag{2.9}$$

It is important to remark that the only requirement on the rest mass distribution of the quasiparticles is that it must be such that $\sum_i B_i(T) < \infty$, viz., a singularity arises only from the lowest mass state. [Note that $0 \leqslant m_0 c^2 C_i(T) \leqslant B_i(T)$ and so $\sum_i B_i(T) < \infty$ implies $\sum_i C_i(T) < \infty$.] Accordingly, the internal energy U has a simple pole, for fixed V, at $T = T_{\max}$ owing to the lowest mass state. Therefore near the singularity we have, from (2.6) and (2.9), that

$$S = U/T_{\max} \tag{2.10}$$

and

$$P = U/3V \tag{2.11}$$

as $T \to T_{\max}$ since $C_0(T) = B_0(T)/3kT$ by (2.4) for $m_0 = 0$. Therefore, the pressure is radiationlike. (Notice that the Helmoholtz free energy $F = kT \sum_i \ln[1 - C_i(T)]$ has a weaker logarithmic singularity at $T = T_{\max}$.)

The entropy (2.10) constitutes the fundamental relation in terms of the energy U and the volume V, via (2.7), since by the usual thermodynamic definitions

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T_{\max}}$$

and (2.12)

$$\frac{P}{T} \equiv \left(\frac{\partial S}{\partial V}\right)_U = \frac{S}{3V}$$

as $T \to T_{\max}$ with the aid of (2.7) thus recovering the pressure (2.11).

In order to apply (2.7) and (2.10) to the interior of a Schwarzschild black hole, one needs to calculate the proper volume inside the surface of the black hole. One expects the interior line element to be spherically symmetric and so[4]

$$ds^2 = e^{\nu(r,t)}\, dt^2 - c^{-2}[e^{\lambda(r,t)}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{2.13}$$

Therefore the proper volume enclosed by the surface with proper area $A = 4\pi R^2$ is

$$V = 4\pi \int_0^R e^{\lambda/2} r^2\, dr \tag{2.14}$$

To proceed further,one needs to know the total stress–energy tensor $T_{\mu\nu}$ and solve Einstein's field equations for $\lambda(r, t)$. Note that by (2.11), the approximate equation of state is $P = \frac{1}{3}\rho c^2$, where P and ρ are the proper pressure and proper density, respectively. However, this allows us to determine $T_{\mu\nu}$ only if one supposes a perfect fluid for the interior of the black hole. However, even in this simple case, one has no exact solution of the field equation of general relativity. [Note that for a static field, if $P = \frac{1}{3}\rho c^2$ is strictly valid, then[4] $P(r) \propto e^{-2\nu(r)}$ and no solution exists with finite radius.] Therefore, in order to determine the proper volume inside the event horizon let us use a very crude estimate based on a static sphere of perfect fluid with constant proper density ρ. This model leads[4] to an exact solution, the interior Schwarzschild solution, for a static spherical distribution of perfect fluid with radius R. Notice that the interior metric matches smoothly[4] to the exterior Schwarzschild metric. However, for finite central pressure, the radius R is restricted by[4] $R > \frac{9}{4}GM/c^2$ and so a smaller spherically symmetric particle with radius $R = 2GM/c^2$ undergoes gravitational collapse. Nonetheless, since exact solutions of the field equation are extremely difficult to find, we shall determine the proper volume with the aid of the interior Schwarzschild solution. Accordingly, we suppose

$$e^{-\lambda} = 1 - r^2/R^2 \tag{2.15}$$

for $r \leqslant R$, with

$$R = 2MG/c^2 \tag{2.16}$$

where G is Newton's gravitational constant. [If one supposes a perfect fluid and $\lambda(r, t)$ is prescribed arbitrarily, as in (2.15) say, then the quantities $\nu(r, t)$, $P(r, t)$, and $\rho(r, t)$ may be determined by the field equations.] Now $e^{-\lambda}$ vanishes for $r = R$ and so the surface $r = R$ is[4] null or lightlike. Consequently, the event horizon with proper area $A = 4\pi R^2$ encloses a finite proper volume

$$V = 4\pi \int_0^R \frac{r^2\,dr}{(1 - r^2/R^2)^{1/2}} = \pi^2 R^3 \tag{2.17}$$

Now, with the aid of (2.7), (2.10), (2.16), and (2.17), the entropy and temperature of a Schwarzschild black hole with $U = Mc^2$ is

$$S_{\mathrm{bh}} = \frac{1}{2\pi}\,(kc^3 A/4\hbar G) \tag{2.18}$$

and

$$T_{\mathrm{bh}} = 4\pi(\hbar c^3/8\pi k G M) \tag{2.19}$$

where $A=4\pi R^2$ is the area of the black hole event horizon. Condition $U/kT \gg 1$ is satisfied for Schwarzschild black holes with masses M much greater than the Planck mass, $M_p \equiv (\hbar c/G)^{1/2} \cong 2 \times 10^{-5}\,g$, since (2.18) and (2.19) give that $U/kT = Mc^2/kT_{bh} = 2(M/M_p)^2$. Note that for a given mass M, the temperature (2.19) is greater than the temperature of the emitted thermal radiation obtained by Hawking,[5] $T_H = \hbar c^3/8\pi kGM$. Also, the entropy (2.18) associated with the internal microstates of the black hole is smaller than the entropy obtained by Hawking,[5] $S_H = kc^3A/4\hbar G$. The entropy S_{ev} evaporated by a Schwarzschild black hole is always greater than or equal[6] to the Hawking entropy S_H, which in turn is strictly greater than the entropy (2.18), viz., $S_{bh} < S_H \leqslant S_{ev}$. Therefore the quantum evaporation[5] of a black hole is an irreversible process. It is interesting that (2.18) is below the conjectured upper bound[7] on the entropy-to-energy ratio for bounded systems, viz., $S \leqslant kc^3A/4\hbar G$.

It should be remarked that from an information-theoretic approach to statistical physics, the entropy (2.18) is a measure of ignorance as to the actual state of the system, viz., a black hole specified only by the external descriptors M, $J=0$, and $Q=0$. Therefore, the ideal-gas nature of the quasiparticles is the means by which this entropy is calculated and in no way assumes the system to be in equilibrium since the situation might be a nonequilibrium one. Accordingly, the entropy S_{bh}, which depends on the mass M of the black hole, is not an equilibrium entropy if M varies with time. For instance, the case of an evaporating black hole where the mass decreases with increasing time. Needless to say, our model does not provide a mechanism for such quantum processes.

3. AREA AND ENTROPY

The identification[7] of black hole surface with black hole entropy suggests

$$S = K_1 R^2 = K_1' V^a U^b \tag{3.1}$$

with the positive constants a and b satisfying

$$3a + b = 2 \tag{3.2}$$

where $V = \pi^2 R^3$, $U = Mc^2 = c^4 R/2G$ and K_1 and K_1' are positive constants. The defining Eq. (2.12) gives

$$T = K_3/R \qquad \text{and} \qquad P = K_2/R^2 \tag{3.3}$$

where

$$K_3 = c^4/2GbK_1 \qquad \text{and} \qquad K_2 = ac^4/2\pi^2 bG \tag{3.4}$$

Given a and K_1 all conceivable thermodynamic information about the black hole is known. Note that the pressure (3.3) is independent of the constant K_1 appearing in the entropy (3.1).

In the determination of the value of the constant K_1, Hawking[5] considers $P \equiv 0$ and $T = \hbar c^3/8\pi kGM$, the Hawking radiation temperature. Hence, by (3.2) to (3.4), $a = 0$, $b = 2$, and $K_1 = \pi c^3 k/\hbar G$. (Note that since $S \sim U^2$, neither the canonical ensemble nor the grand-canonical ensemble exists.) If, however, we require the pressure to be a radiation pressure, viz., $P = U/3V$, then (3.4) gives $3a = b$ and so by (3.2), $a = 1/3$ and $b = 1$. Our model for a Schwarzschild black hole, determined by (2.7) and (2.10), gives, in addition, that $K_1 = kc^3/2\hbar G$. Note that for $0 < b \leqslant 1$, both the canonical and grand-canonical ensembles exist. (It is quite possible to choose a value for K_1 in order to obtain the Hawking temperature in addition to having a radiation pressure, viz., $K_1 = 2\pi c^3 k/\hbar G$, $a = 1/3$, and $b = 1$. This latter choice gives twice the Hawking entropy, $S = kc^3 A/2\hbar G$.)

It should be remarked that the entropy (2.18) is a result of taking a trace of the density matrix associated with the distinguishable quasiparticles constituting the black hole with external state described by M, $Q = 0$, and $J = 0$. For a nonstationary black hole—quantum processes give rise to black hole evaporation[5]—these parameters vary with time and additional external parameters may exist.

4. SUPERADDITIVITY AND CONCAVITY

The relativistic gas model of the black hole interior of Section 2 is a microscopic model which satisfies the general macroscopic results[1] summarized in Section 1. In fact, our statistical model is of the type $(\mathbf{S}, \bar{\mathbf{H}}, \bar{\mathbf{C}})$. The proof of this assertion is based entirely on our explicit result for the entropy $S(U, V)$ obtained by eliminating $T_{\max}$ between Eqs. (2.7) and (2.10).

The entropy (2.10) is strictly superadditive

$$S(U_1 + U_2, V_1 + V_2) > S(U_1, V_1) + S(U_2, V_2) \tag{4.1}$$

since by (2.7), $T_{\max}(V_1 + V_2) < T_{\max}(V_i)$ for $i = 1, 2$ and $V_i \neq 0$. The nonextensivity of (2.10), together with (1.4) and (4.1), implies the failure of (1.3). That concavity is violated follows also directly from (2.10) since

$$S(\lambda U, \lambda V) = \lambda^{4/3} S(U, V) \leqslant \lambda S(U, V) \tag{4.2}$$

when $0 \leqslant \lambda \leqslant 1$. [Note that $S(0) = 0$.] The Gibbs free energy

$$G \equiv U + PV - TS = PV \tag{4.3}$$

by using (2.10). Hence with the aid of (2.7)

$$C_P \equiv -T\left(\frac{\partial^2 G}{\partial T^2}\right)_P = -12PV/T_{\max} < 0 \tag{4.4}$$

Similarly, $C_V \equiv -T(\partial^2 F/\partial T^2)_V = 0$, $K_T \equiv -(1/V)(\partial^2 G/\partial P^2)_T = 0$, and $K_S = -(1/V)(\partial V/\partial P)_S = 3/4P > 0$. It is important to remark that despite a negative specific heat, the system is not thermally unstable since the state of the system is not determined by entropy maximization.[1] In fact, on merging two such systems, with extensive variables U and V, the final entropy increases by (4.1). Notice that for Schwarzschild black holes, the energy $U = Mc^2$ is an extensive variable and so by (2.16) the volume V enclosed by the event horizon, viz., $V = \pi^2 R^3$, is not an extensive variable. Therefore, for Schwarzschild black holes (2.10) gives

$$S(U_1, V_1) + S(U_2, V_2) \leqslant S(U_f, V_f) \tag{4.5}$$

where $U_f = U_1 + U_2$ and $V_f = [V_1^{1/3} + V_2^{1/3}]^3$. Consequently, the coalescence of two Schwarzschild black holes always gives rise to an increase of the entropy.

If one considers two systems A and B with entropy (2.10), then the total entropy is

$$S \equiv S_A + S_B = \pi^{-2/3}(k/\hbar c)[U_A V_A^{1/3} + (U - U_A)\, V_B^{1/3}] \tag{4.6}$$

where $U = U_A + U_B$ and a symmetric function $f(V_A, V_B) = f(V_B, V_A) =$ const, which serves as a constraint, is twice differentiable with respect to its two arguments V_A, V_B. [For a gas $f(V_A, V_B) = V_A + V_B = V$ and for a Schwarzschild black hole $f(V_A, V_B) = V_A^{1/3} + V_B^{1/3} = V^{1/3}$.] The stationary value or extremum of (4.6) is given by

$$U_A = U_B = U/2 \qquad \text{and} \qquad V_A = V_B \tag{4.7}$$

However, the extremum cannot be a maximum since $\partial^2 S/\partial U_A^2 = 0$ and so

$$\begin{vmatrix} \dfrac{\partial^2 S}{\partial U_A^2} & \dfrac{\partial^2 S}{\partial U_A\, \partial V_A} \\ \dfrac{\partial^2 S}{\partial V_A\, \partial U_A} & \dfrac{\partial^2 S}{\partial V_A^2} \end{vmatrix} < 0.$$

This result is just as for the generalized ideal gas[1] with $h < g + 1$.

The nonextensive entropy (2.10) is the result of short-range, many-body forces[3] among the constituents of supermassive particles. The confining forces among (Fermi or Bose) constituents are purely a manifestation of the correlations resulting from the strictly classical statistics of the quasiparticles. The potential energy $u(\mathbf{r}_1,..., \mathbf{r}_n)$ of n_1 particles of mass m_1, n_2 particles of mass m_2, etc. located at $\mathbf{r}_1 \cdots \mathbf{r}_n \in V$ is given by[3]

$$V^n n_1!\, n_2! \cdots = \int_V d\mathbf{r}_1 \cdots \int_V d\mathbf{r}_n \exp[-u(\mathbf{r}_1,..., \mathbf{r}_n)/kT] \tag{4.8}$$

with $n = n_1 + n_2 + \cdots$. Suppose $u(\mathbf{r}_1,..., \mathbf{r}_n) \geqslant -A$, then since $\ln n_1! + \ln n_2! \cdots \leqslant \ln n! \leqslant n \ln n$, we have from (4.8) that

$$u(\mathbf{r}_1,..., \mathbf{r}_n) \geqslant -Bn \ln n \tag{4.9}$$

with $B = kT$. Hence, the interaction $u(\mathbf{r}_1,..., \mathbf{r}_n)$ violates the stability condition[8] and so the system does not possess a thermodynamic limit. Of course, nonextensivity is a consequence of the nonexistence of the thermodynamic limit.

5. CONCLUSION

Black hole entropy is shown to be proportioned to the area. The internal configurations of the black hole are described by means of distinguishable quasiparticles and so the entropy follows from taking the trace of the density matrix of a noninteracting gas. This ideal gas of distinguishable quasiparticles is equivalent to an interacting gas of particles satisfying Maxwell–Boltzmann statistics. The potential energy of n such particles violates the stability condition, viz., no thermodynamic limit, and so nonextensivity ensues.

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